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UNSTEADY MOTION OF A MAXWELLIAN FLUID DROPLET IN A MAXWELLIAN MEDIUM UNDER THE ACTION OF MONOTONIC AND PERIODIC FORCES

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Unsteady motion of a Maxwellian fluid droplet, which arises in a quiescent Maxwellian medium under the action of monotonic and periodic forces, is considered. In the initial period of time smaller than the relaxation time, the droplet is affected by elastic forces on the part of the fluid; moreover, the droplet itself is a viscoelastic material. A solution of the problem in the first approximation is found. The dependence of the amplitude of droplet velocity and the shift of the phase of oscillations on the relaxation time of the external and internal media and also on the frequency of oscillations of the driving force is analyzed. The passage to the limit in terms of density and viscosity of the internal medium is performed.

Key words: *Maxwellian fluid, viscoelastic medium, relaxation time, periodic forces.*

Introduction. The motion of liquid droplets, solid particles, and gas bubbles in a viscous incompressible fluid has been considered in many papers. The behavior of droplets and bubbles under the action of thermocapillary, buoyancy, and periodic forces has been studied in much detail (see, e.g., [1, 2]). It seems of interest to consider the droplet behavior in other, more complicated media. In the present work, we consider the motion of a droplet, a bubble, and a solid sphere in a viscoelastic medium. In particular, the case of a Maxwellian fluid or a Maxwellian body is described.

Mathematically, the Maxwellian model is described by an equation of state of the form [3]

$$T_{\rm rel}\,\frac{\tilde{d}P}{dt}+P=-pI+2\mu D,$$

where P is the stress tensor, $\tilde{d}P/dt = dP/dt + P\cdot W + (P\cdot W)^{\rm t}$, $W = (\nabla\boldsymbol{v} - \nabla\boldsymbol{v}^{\rm t})/2$ is the Jaumann derivative providing invariance with respect to rotation, D is the strain-rate tensor, p is the pressure, t is the time, and v is the velocity. The magnitude of the relaxation time $T_{\rm rel}$, which determines the behavior and properties of the medium, is important here. If the test duration is rather short ($t \ll T_{\rm rel}$), the Maxwellian body behaves as the Hooke's solid; for $T_{\rm rel} \ll t$, the material is the Newtonian fluid.

This property of the Maxwellian medium allows one to use this model to study amazing phenomena similar to those observed in the experiments of [4, 5], where the behavior of two spherical disperse elements (droplets in a low-viscosity liquid matrix, solid particles in a high-viscosity liquid matrix, or air voids in a viscoelastic gel) was considered. It was found that two or more disperse elements being completely isolated from external force, temperature, and concentration fields and being located at distances of the order of their size, are mutually attracted until a complete contact (coagulation) occurs. As was demonstrated in the experiments of [6, 7], if the shear stresses are very low ($\tau < \tau_*$), polar fluids behave as elastic bodies with a certain yield strength τ_*, and a transition to the plasticity mode is observed at $\tau > \tau_*$. Thus, the medium first behaves as an elastic body and then passes to the state of a Maxwellian body. With allowance for this fact, we can formulate the following problem. A viscoelastic

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Maxwellian medium contains a spherical disperse element, and a certain body force is acting on this element. The present study is aimed at determining the velocity and pressure fields.

In the general case, the fluid viscosity is a variable quantity; as the relaxation time is $T_{\rm rel} = \mu/G$, where G is the dynamic shear modulus and μ is the viscosity, it is also a time-dependent quantity. Thus, these characteristics of the medium can take different values. Yet, these parameters remain almost unchanged if the strain rate is rather low, and they can be considered as constants.

The shear stress relaxes with time, and the Maxwellian model starts to describe a conventional Newtonian fluid. To avoid this, one can reasonably consider another, nonmonotonic type of driving forces to study the Maxwellian medium properties. The present paper describes the solution of the problem of motion of a Maxwellian fluid droplets in a viscoelastic medium under the action of periodic forces. A necessary condition is commensurable values of the relaxation time of the medium and the period of oscillations. For many viscoelastic bodies, the value of $T_{\rm rel}$ lies within the interval from 10^{-3} to 10^{-1} sec; hence, the oscillation frequency in this case should be about 10^1 to 10^3 Hz. Low values of oscillation frequency do not allow one to detect shear elasticity, and the equation of state will be too cumbersome in terms of mathematics.

Information about the Maxwellian fluid properties can be found in [3, 8, 9].

1. Formulation of the Problem. We have to find a surface Γ_t that divides the space $\mathbb{R}^3$ into a bounded simply connected domain Ω^+ and its supplement $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$, the fields of velocities $\boldsymbol{v}$ and pressures p depending on the time t and spatial coordinates $\boldsymbol{x}$ and satisfying the system of equations

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = \frac{1}{\rho}\,\mathrm{div}\, P + \boldsymbol{g}, \qquad \nabla \cdot \boldsymbol{v} = 0,$$
$$T_{\rm rel}\,\frac{\tilde{d}P}{dt} + P = -pI + 2\mu D, \tag{1}$$

the conjugation conditions

$$[P \cdot \boldsymbol{n}]^{\pm} = \sigma K \boldsymbol{n},$$
$$V_n = \boldsymbol{v} \cdot \boldsymbol{n}, \qquad [\boldsymbol{v}]^{\pm} = 0 \quad \text{on} \quad \Gamma_t,$$

the condition at infinity

$$\boldsymbol{v} \to 0 \quad \text{as} \quad |\boldsymbol{x}| \to \infty$$

and the initial conditions

$$\boldsymbol{v} = 0, \quad \Gamma_t = \Gamma_0 = \{\boldsymbol{x}\colon |\boldsymbol{x}| = a\}, \quad P = P_0 \quad \text{for} \quad t = 0.$$

Here, the density ρ and viscosity $\mu = \rho\nu$ are piecewise-constant functions with a discontinuity surface Γ_t, ν is the kinematic viscosity, σ is the surface-tension coefficient, P is the stress tensor, $D(\boldsymbol{v})$ is the strain-rate tensor, K is the half-sum of the principal curvatures of the surface Γ_t (trace of the curvature tensor), V_n is the velocity of motion of the surface Γ_t along the normal $\boldsymbol{n}$ external to Ω^+, and $[f]^{\pm} = f^+ - f^-$ ($f^{\pm}$ are the limiting values of the function $f(x,t)$ for $\boldsymbol{x}$ tending from $\Omega^{\pm}$ to a point of the surface Γ, respectively).

It is seen from the boundary conditions that the velocity fields are continuous when passing through Γ_t, whereas the fields of pressure and shear stresses have a discontinuity on this surface. As a result, external body forces initiate the droplet motion. The density of these forces $\boldsymbol{g}(t) = (0, 0, g(t))$ is prescribed, and we assume that $g(0) = 0$. The problem of droplet acceleration in a viscous fluid under the action of thermocapillary and body forces was solved in [2].

2. Simplifying Assumptions. We pass to a coordinate system fitted to the center of mass of the droplet moving in the original system with a velocity $\boldsymbol{u}(t) = (0, 0, u(t))$, i.e.,

$$\boldsymbol{x}' = \boldsymbol{x} - \int\limits_0^t \boldsymbol{u}(t)\,dt, \qquad t' = t.$$

We introduce new unknown functions

$$\boldsymbol{v}' = \boldsymbol{v} - \boldsymbol{u}, \qquad P' = P + \rho\boldsymbol{x}'(\boldsymbol{g} - \dot{\boldsymbol{u}}(t))I,$$

$$p' = p - \rho\boldsymbol{x}'[\boldsymbol{g} + T_{\rm rel}\dot{\boldsymbol{g}} - \dot{\boldsymbol{u}}(t) - T_{\rm rel}\ddot{\boldsymbol{u}}(t)].$$

In the new variables, system (1) transforms to a system of the following form:

$$\frac{\partial \boldsymbol{v}'}{\partial t'} + \boldsymbol{v}' \cdot \nabla \boldsymbol{v}' = \frac{1}{\rho}\operatorname{div} P', \qquad \nabla \cdot \boldsymbol{v}' = 0,$$

$$T_{\rm rel}\Big\{\frac{\partial P'}{\partial t'} + \boldsymbol{v}' \cdot \nabla P' - \rho \boldsymbol{v}'(\boldsymbol{g} - \dot{\boldsymbol{u}}(t)) + (P' - \rho \boldsymbol{x}'(\boldsymbol{g} - \dot{\boldsymbol{u}}(t))) \cdot W \tag{2}$$

$$+ ((P' - \rho \boldsymbol{x}'(\boldsymbol{g} - \dot{\boldsymbol{u}}(t))) \cdot W)^{\rm t}\Big\} + P' = -p'I + 2\mu D'.$$

It is important to note that the first equation of system (2) is invariant with respect to this transformation.

We choose the quantities a, a^2/ν^-, $a^2 g_0/\nu^-$, and $\rho^- a g_0$ as the scales of length, time, velocity, and pressure, respectively, where a is the droplet radius at the initial time and g_0 is a certain "mean" acceleration. The dimensionless equations of motion with omitted primes acquire the form

$$\frac{\partial \boldsymbol{v}}{\partial t} + M\boldsymbol{v} \cdot \nabla \boldsymbol{v} = \frac{\rho^-}{\rho}\operatorname{div} P, \qquad \nabla \cdot \boldsymbol{v} = 0,$$

$$T\Big\{\frac{\partial P}{\partial t} + M\Big(\boldsymbol{v} \cdot \nabla P - \frac{\rho}{\rho^-}\boldsymbol{v}(\boldsymbol{g} - \dot{\boldsymbol{u}}(t)) + (P - \rho \boldsymbol{x}(\boldsymbol{g} - \dot{\boldsymbol{u}}(t))) \cdot W$$

$$+ ((P - \rho \boldsymbol{x}(\boldsymbol{g} - \dot{\boldsymbol{u}}(t))) \cdot W)^{\rm t}\Big)\Big\} + P = -pI + 2\frac{\mu}{\mu^-}D,$$

where $M = a^3 g_0/(\nu^-)^2$ and $T = (\nu^-/a^2)T_{\rm rel}$.

We assume that $a^3 g_0 \ll (\nu^-)^2$ and $\rho^- a g_0 \ll \sigma/a$. The first relation ensures the smallness of the parameter M, and the second one is necessary for capillary forces to significantly prevail over pressure. In this case, the droplet can retain an almost spherical shape. Formally decomposing the functions $\boldsymbol{v}$, p, and P into series in terms of M, we obtain the following problem with $M = 0$ as the first approximation:

$$\frac{\partial \boldsymbol{v}}{\partial t} = \frac{\rho^-}{\rho}\operatorname{div} P, \qquad \nabla \cdot \boldsymbol{v} = 0,$$

$$T\frac{\partial P}{\partial t} + P = -pI + 2\frac{\mu}{\mu^-}D,$$

$$[P \cdot \boldsymbol{n}]^{\pm} = 2\sigma/(a g_0) \cdot \boldsymbol{n}, \tag{3}$$

$$\boldsymbol{v}^+ \cdot \boldsymbol{n} = 0, \qquad \boldsymbol{v}^- \cdot \boldsymbol{n} = 0, \qquad \boldsymbol{v}^+ \cdot \boldsymbol{\tau} = \boldsymbol{v}^- \cdot \boldsymbol{\tau} \quad \text{on} \quad \Gamma,$$

$$\boldsymbol{v} + \boldsymbol{u} \to 0, \qquad |\boldsymbol{x}| \to \infty,$$

$$\boldsymbol{v} = 0, \quad \boldsymbol{u} = 0, \quad \dot{\boldsymbol{u}} = 0, \qquad t = 0.$$

This problem admits an exact solution with a spherical interface $\Gamma_t = \Gamma = \{\boldsymbol{x}\colon\ |\boldsymbol{x}| = 1\}$. Here, $\boldsymbol{\tau}$ is a vector tangential to Γ.

We determine the Laplace transformation by the formula

$$A^*(s) = \int_0^\infty A(t)\, e^{-st}\, dt$$

and apply it to each equation in system (3). By virtue of initial data, we obtain

$$s\boldsymbol{v}^* = (\rho^-/\rho)\operatorname{div} P^*; \tag{4}$$

$$(Ts + 1)P^* = -p^* I + 2(\mu/\mu^-)D^*. \tag{5}$$

Substituting (5) into (4) and denoting $\alpha^{\pm} = 1/(T^{\pm}s+1)$, where T^+ and T^- are the relaxation times for the internal and external fluids, respectively, we obtain the following equation:

$$s\boldsymbol{v}^* = \alpha(\rho^-/\rho)(-\nabla p^* I + (\mu/\mu^-)\Delta \boldsymbol{v}^*). \tag{6}$$

The conjugation conditions for P become

$$[(-\alpha^+ p^+ + \alpha^- p^-)^* - (\rho^0 - 1)x_3(g^* - su^*)]\boldsymbol{n} + 2(\alpha^+\mu^0 D^*(\boldsymbol{v}^+)\cdot\boldsymbol{n} - \alpha^- D^*(\boldsymbol{v}^-)\cdot\boldsymbol{n}) = 2\sigma/(ag_0 s)\cdot\boldsymbol{n},$$

where $\rho^0 = \rho^+/\rho^-$, $\nu^0 = \nu^+/\nu^-$, and $\mu^0 = \rho^0\nu^0$.

Let (r, φ, θ) be a spherical coordinate system. We seek for the solution under the assumption of axial symmetry. We introduce the function $\psi^*(r, \theta, s)$ by the equalities

$$v_r^* = -\frac{1}{r^2\sin\theta}\,\frac{\partial\psi^*}{\partial\theta}, \qquad v_\theta^* = \frac{1}{r\sin\theta}\,\frac{\partial\psi^*}{\partial r}.$$

Then, system (6) is written as

$$\alpha\,\frac{\rho^-}{\rho}\,\frac{\partial p}{\partial r} = \frac{1}{r^2}\,\frac{\partial}{\partial\xi}\Big\{\frac{\nu}{\nu^-}\,\alpha E^2\psi^* - s\psi^*\Big\},$$

$$\alpha\,\frac{\rho^-}{\rho}\,\frac{\partial p}{\partial\xi} = -\frac{1}{1-\xi^2}\,\frac{\partial}{\partial r}\Big\{\frac{\nu}{\nu^-}\,\alpha E^2\psi^* - s\psi^*\Big\},$$

where $E^2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{1-\xi^2}{r^2}\,\dfrac{\partial^2}{\partial\xi^2}$ and $\xi = \cos\theta$. Correspondingly, the components of the transformed stress tensor in terms of ψ^* have the form

$$P_{r\theta}^* = -\frac{\mu}{\mu^-}\,\frac{\alpha}{(1-\xi^2)^{1/2}}\Big[E^2\psi^* - 2r\,\frac{\partial}{\partial r}\Big(\frac{1}{r}\,\frac{\partial\psi^*}{\partial r}\Big)\Big],$$

$$\frac{\partial}{\partial\xi}\,P_{rr}^* = \frac{\mu}{\mu^-}\,\frac{\partial}{\partial r}\Big[\frac{1}{1-\xi^2}\Big(\alpha E^2\psi^* - \frac{\nu^-}{\nu}\,s\psi^*\Big) + \frac{2\alpha}{r^2}\,\frac{\partial^2\psi^*}{\partial\xi^2}\Big].$$

As a result, we have the following problem for the functions ψ^* and u^*:

$$\begin{aligned} &E^2[\nu^0\alpha^+E^2\psi^* - s\psi^*] = 0 \qquad \text{for} \quad r < 1,\\ &E^2[\alpha^-E^2\psi^* - s\psi^*] = 0 \qquad \text{for} \quad r > 1; \end{aligned} \tag{7}$$

$$\begin{aligned} &\psi^{*+} = 0, \qquad \psi^{*-} = 0, \qquad \psi_r^{*+} = \psi_r^{*-},\\ &\alpha^0\mu^0(\psi_{rr}^* - 2\psi_r^*)^+ - (\psi_{rr}^* - 2\psi_r^*)^- = 0 \qquad \text{for} \quad r = 1; \end{aligned} \tag{8}$$

$$\frac{\psi_r^*}{r} \to u^*(1-\xi^2), \qquad \frac{\psi_\xi^*}{r^2} \to -u^*\xi \qquad \text{as} \quad r \to \infty; \tag{9}$$

$$\begin{aligned} &(\rho^0 - 1)(su^* - g^*) + \mu^0\,\frac{\partial}{\partial r}\Big\{\frac{\alpha E^2\psi^* - {\nu^0}^{-1}s\psi^*}{1-\xi^2} + \frac{2}{r^2}\,\alpha\psi_{\xi\xi}^*\Big\}^+\\ &-\frac{\partial}{\partial r}\Big\{\frac{\alpha E^2\psi^* - s\psi^*}{1-\xi^2} + \frac{2}{r^2}\,\alpha\psi_{\xi\xi}^*\Big\}^- = 0 \qquad \text{for} \quad r = 1. \end{aligned} \tag{10}$$

Equation (10) was obtained by differentiating the dynamic condition in terms of ξ. This does not expand the class of solutions because the pressure outside and inside the droplet is determined from the stream function with accuracy to an additive constant.

3. Solution of the Problem. We seek for the solution in the form

$$\psi^*(r, \xi, s) = rf^*(r, s)(1-\xi^2).$$

Correspondingly, we rewrite problem (7)–(10) in terms of an operator L such that $L^2 = \dfrac{1}{r^2}\,\dfrac{\partial}{\partial r}\Big(r^2\dfrac{\partial}{\partial r}\Big) - \dfrac{2}{r^2}$:

$$\begin{aligned} &L^2[\nu^0\alpha L^2 f^* - sf^*] = 0 \qquad \text{for} \quad r < 1,\\ &L^2[\alpha L^2 f^* - sf^*] = 0 \qquad \text{for} \quad r > 1; \end{aligned} \tag{11}$$

$$r = 1: \qquad f^{*+} = 0, \quad f^{*-} = 0, \quad f_r^{*+} = f_r^{*-}, \quad \alpha^0\mu^0 f_{rr}^{*+} - f_{rr}^{*-} = 0; \tag{12}$$

$$r \to \infty: \qquad f_r^* \to u^*/2, \qquad f^*/r \to u^*/2; \tag{13}$$

$$r = 1: \qquad (1-\rho^0)(su^* - g^*) + [\alpha f_{rrr}^* + \alpha f_{rr}^* - (s+6\alpha) f_r^*]^- = \mu^0[\alpha f_{rrr}^* + \alpha f_{rr}^* - (s/\nu^0 + 6\alpha) f_r^*]^+. \tag{14}$$

From the integral identity

$$\int_0^1 (L^2\omega) r^3\, dr = r^2(r\omega_r - \omega)_0^1$$

with the function $\omega = L^2 f^* - (s/(\nu^0\alpha)) f^*$, we can readily find that the right side of Eq. (14) equals zero. Thus, condition (14) is simplified:

$$(1-\rho^0)(su^* - g^*) + \{\alpha f_{rrr}^* + \alpha f_{rr}^* - (s+6\alpha) f_r^*\}^- = 0 \qquad \text{at} \quad r = 1. \tag{15}$$

The parameter L^2 obeys the relation

$$L^2\varphi = (1/r)\pounds^2 r\varphi,$$

where $\pounds^2 = \partial^2/\partial r^2 - 2/r^2$. Hence, the equation

$$L^2[\nu\alpha L^2 f^* - sf^*] = 0$$

can be rewritten as

$$\pounds^2[\nu\alpha\pounds^2 rf^* - srf^*] = 0.$$

Its general solution has the form

$$f^* = c_1 r + \frac{c_2}{r^2} + \frac{1}{r}\Big[c_3\, e^{rb}\Big(b - \frac{1}{r}\Big) + c_4\, e^{-rb}\Big(b + \frac{1}{r}\Big)\Big],$$

where $b^2 = s/(\nu\alpha)$. After simple transformations, with allowance for boundedness of the velocity field at $r = 0$ and condition (13), the solution of Eqs. (11) can be presented in the form

$$f^*(r,s) = C_1 F(\beta r) + C_2 r, \qquad r < 1,$$

$$f^*(r,s) = u^* r/2 + C_3/r^2 + C_4 G(\gamma r), \qquad r > 1,$$

where $F(z) = d(\sinh z/z)/dz$, $G(z) = d(e^{-z}/z)/dz$, $\beta = \sqrt{s/(\nu^0\alpha^+)}$, and $\gamma = \sqrt{s/\alpha^-}$. The functions $C_1(s), \ldots, C_4(s)$ are determined from Eqs. (12). As a result, we obtain

$$f^*(r,s) = \frac{3(1+\gamma)u^*(s)/2}{3+\gamma+\mu^0 H(\beta)}\,\frac{F(\beta r) - F(\beta) r}{\beta F'(\beta) - F(\beta)}, \qquad r < 1,$$

$$f^*(r,s) = -\frac{3(2+\mu^0 H(\beta))u^*(s)/2}{3+\gamma+\mu^0 H(\beta)}\, e^{\gamma}\Big(G(\gamma r) - \frac{G(\gamma)}{r^2}\Big) + \frac{1}{2}u^*\Big(r - \frac{1}{r^2}\Big), \quad r > 1.$$

Here, $H(z) = \dfrac{z^2 F''(z)}{zF'(z) - F(z)} = \dfrac{z(z^2+6) - 3(z^2+2)\tanh z}{(z^2+3)\tanh z - 3z}$. Then, from Eq. (15), after cumbersome calculations, we find

$$u^*(s) = \frac{(\rho^0 - 1)g^*(s)}{(1/2+\rho^0)s + \alpha^- B^*(s)}, \tag{16}$$

where

$$B^*(s) = \frac{3}{2}(2+\mu^0 H(\beta))C^*(s), \qquad C^*(s) = \frac{3(1+\sqrt{s/\alpha^-})}{3+\sqrt{s/\alpha^-}+\mu^0 H(\beta)}.$$

The following asymptotic formulas are valid:

$$H(z) = 3 + z^2/7 + O(z^4), \qquad z \to 0,$$

$$H(z) = z + 3/z + O(1/z^3), \qquad z \to \infty,$$

$$B^*(0)=\frac{3(2+3\mu^0)}{2(1+\mu^0)},\qquad B^*(s)\approx\frac{9\rho^0\sqrt{\nu^0}T^0}{2(1+\rho^0\sqrt{\nu^0}T^0)}\sqrt{s(T^-s+1)},\quad s\to\infty$$

($T_0=T^+/T^-$). The function $B^*(s)$ can be represented as $B^*(s)=B^*(0)+sb^*(s)$, where the principal terms in the expansion $b^*(s)$ in the vicinity of zero and infinity have the form

$$b^*(s)\approx\frac{1}{2}\Big(\frac{2+3\mu^0}{1+\mu^0}\Big)^2\sqrt{\frac{T^-s+1}{s}},\qquad s\to 0,$$

$$b^*(s)\approx\frac{9\rho^0\sqrt{\nu^0}T^0}{2(1+\rho^0\sqrt{\nu^0}T^0)}\sqrt{\frac{T^-s+1}{s}},\qquad s\to\infty.$$

We rewrite Eq. (16) in the form

$$(1/2+\rho^0)(T^-s+1)su^*+B^*(0)u^*+sb^*(s)u^*=(\rho^0-1)(T^-s+1)g^*(s).$$

Obviously, $su^*(s)$ is the image of $u'(t)$. The expression $sb^*(s)u^*(s)$ is the image of $\int_0^t u'(t_1)b(t-t_1)\,dt_1$. Indeed, we have

$$sb^*(s)u^*(s)=s\int_0^\infty b(t)\,\mathrm{e}^{-st}\,dt\int_0^\infty u(t_1)\,\mathrm{e}^{-st_1}\,dt_1$$

$$=\int_0^\infty b(t-t_1)\,\mathrm{e}^{-s(t-t_1)}\,d(t-t_1)\int_0^\infty u'(t_1)\,\mathrm{e}^{-st_1}\,dt_1$$

$$=\int_0^\infty\int_0^t b(t-t_1)u'(t_1)\,\mathrm{e}^{-s(t-t_1)}\,\mathrm{e}^{-st_1}\,dt_1\,dt=\int_0^\infty\int_0^t b(t-t_1)u'(t_1)\,dt_1\,\mathrm{e}^{-st}\,dt.$$

Thus, Eq. (16) yields the integrodifferential equation

$$\Big(\frac{1}{2}+\rho^0\Big)[T^-u''(t)+u'(t)]+B^*(0)u(t)+\int_0^t u'(t_1)b(t-t_1)\,dt_1=(\rho^0-1)(T^-g'(t)+g(t)). \qquad (17)$$

Directing T^+ and T^- to zero, we use Eq. (16) or (17) to obtain a velocity equal to that for the Newtonian fluid [2].

Let the function $g(t)$ have a limit as $t\to\infty$. Then, by virtue of the equality

$$\lim_{t\to\infty}u(t)=\lim_{s\to 0}su^*(s)$$

Eq. (16) yields the formula for the limiting velocity:

$$\lim_{t\to\infty}u(t)=\lim_{s\to 0}\frac{(\rho^0-1)sg^*(s)}{(1/2+\rho^0)s+\alpha^-B^*(s)}=\frac{2(1+\mu^0)}{3(2+3\mu^0)}(\rho^0-1)\lim_{t\to\infty}g(t).$$

The resultant expression exactly coincides with the velocity of droplet motion under the action of buoyancy forces, which is obtained by the Hadamard–Rybchinskii formula [10]. Naturally, this result is independent of the relaxation time $T_{\rm rel}$. This agrees with the physical laws that describe the behavior of the Maxwellian fluid.

4. Transition to the Differential Equation. In some limiting cases, we can pass from Eqs. (16), (17) to the differential equation. We consider the motion of a gas bubble in the Maxwellian fluid in the case the system experiences some impact or push. Let $\mu^0=0$ and $g(t)=t\,\mathrm{e}^{-t}$, which corresponds to $g^*(s)=1/(s+1)^2$. Equation (16) takes the form

$$[(1/2+\rho^0)s(T^-s+1)(3+\sqrt{s(T^-s+1)}\,)+9(1+\sqrt{s(T^-s+1)}\,)]u^*(s)$$

$$=(T^-s+1)(3+\sqrt{s(T^-s+1)}\,)(\rho^0-1)/(s+1)^2\equiv q^*(s).$$

Multiplying both parts of the equality by

$$r^*(s)=(1/2+\rho^0)s(T^-s+1)(\sqrt{s(T^-s+1)}-3)+9(\sqrt{s(T^-s+1)}-1)$$

and introducing the notation

$$P^*(s) = (1/2+\rho^0)^2 s^2 (T^- s+1)^2 (s(T^- s+1)-9)$$

$$+ 18(1/2+\rho^0)s(T^- s+1)(s(T^- s+1)-3) + 81(s(T^- s+1)-1)$$

or

$$P^*(s) = (1/2+\rho^0)^2 (T^-)^3 s^6 + 3(1/2+\rho^0)^2(T^-)^2 s^5 + (18(1/2+\rho^0)(T^-)^2 + 3(1/2+\rho^0)^2 T^-$$

$$-9(1/2+\rho^0)^2(T^-)^2)s^4 + (36(1/2+\rho^0)T^- + (1/2+\rho^0)^2 - 18(1/2+\rho^0)^2 T^-)s^3$$

$$+(-9(1/2+\rho^0)^2 + 9 + 18\rho^0 - 54(1/2+\rho^0)T^- + 81T^-)s^2 + (54-54\rho^0)s - 81,$$

we obtain

$$P(s)u^*(s) = F^*(s), \qquad F^*(s) = r^*(s)q^*(s).$$

Finally, we can readily pass to the sixth-order differential equation

$$P\Big(\frac{d}{dt}\Big)u(t) = F(t),$$

where $F(t)$ is the original of the image $F^*(s)$. A question arises on the initial data for deriving this equation. They can be determined using the inverse Laplace transform for decomposing the function $u^*(s)$ into a Taylor series at infinity. We have

$$u^*(s) = \frac{a_3}{s^3} + \frac{a_4}{s^4} + \frac{a_5}{s^5} + \frac{a_6}{s^6} + O\Big(\frac{1}{s^7}\Big),$$

where

$$a_3 = \frac{\rho_0 - 1}{1/2+\rho_0}, \qquad a_4 = \frac{-2\rho_0 + 2}{1/2+\rho_0}, \qquad a_5 = \frac{3\rho_0 - 3 - 18(\rho_0-1)/((1+2\rho_0)T^-)}{1/2+\rho_0},$$

$$a_6 = \frac{2(\rho_0-1)(36/\sqrt{T^-} + 18/T^- - 8T^-(1+2\rho_0) + 36)}{(1+2\rho_0)^2 T^-}.$$

We denote the inverse Laplace transform by Z^{-1}. Then, we have $Z^{-1}[1/s^n] = t^{n-1}/(n-1)!$, whence we obtain $u(t) = a_3 t^2/2 + a_4 t^3/6 + a_5 t^4/24 + a_6 t^5/120 + O(t^6)$. As a result, we determine the additional initial data

$$u''(0) = a_3, \qquad u'''(0) = a_4, \qquad u^{IV}(0) = a_5, \qquad u^{V}(0) = a_6.$$

Similar reduction is performed for a solid sphere with $\mu^0 = \infty$. In this case, Eq. (16) acquires the form

$$[(1/2+\rho^0)s(T^- s+1) + (9/2)(1+\sqrt{s(T^- s+1)}\,)]u^*(s) = (T^- s+1)(\rho^0-1)/(s+1)^2 \equiv q^*(s).$$

The regularized symbol is the expression $r^*(s) = (1/2+\rho^0)s(T^- s+1) - (9/2)(\sqrt{s(T^- s+1)} - 1)$.

5. Motion under the Action of Periodic Body Forces. Now we consider the problem of droplet motion arising under the action of periodic forces. For instance, such an action is very important inside a spacecraft in the absence of gravity. Problems of this type are called "g-jitter" problems.

We assume that $g(t) = A\,\mathrm{e}^{i\omega t}$. In this case, the functions $\boldsymbol{u}$, $\boldsymbol{v}$, P, and p can be found in the form

$$\boldsymbol{u} = \mathrm{Re}\,\hat{\boldsymbol{u}}\,\mathrm{e}^{i\omega t}, \qquad \boldsymbol{v} = \mathrm{Re}\,\hat{\boldsymbol{v}}\,\mathrm{e}^{i\omega t}, \qquad P = \mathrm{Re}\,\hat{P}\,\mathrm{e}^{i\omega t}, \qquad p = \mathrm{Re}\,\hat{p}\,\mathrm{e}^{i\omega t}.$$

Substituting these expressions into the system of equations of motion and state (3) and omitting the hats, we obtain the following equations:

$$i\omega\boldsymbol{v} = (\rho^-/\rho)\,\mathrm{div}\,P; \qquad (18)$$

$$(i\omega T + 1)P = -pI + 2(\mu/\mu^-)D. \qquad (19)$$

Substituting (19) into (18) and denoting $\hat{\alpha}^{\pm} = 1/(T^{\pm}s+1)$, we obtain

$$i\omega\boldsymbol{v} = \hat{\alpha}(\rho^-/\rho)(-\nabla pI + (\mu/\mu^-)\Delta\boldsymbol{v}).$$

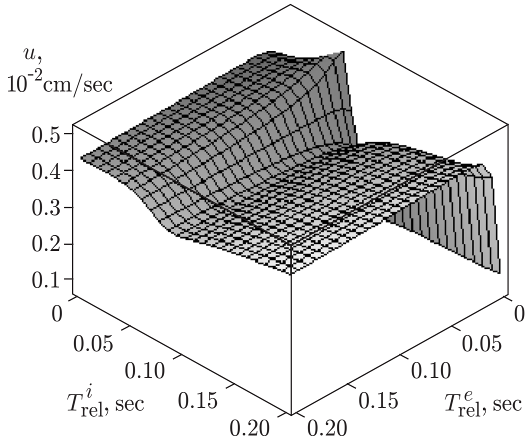

Fig. 1

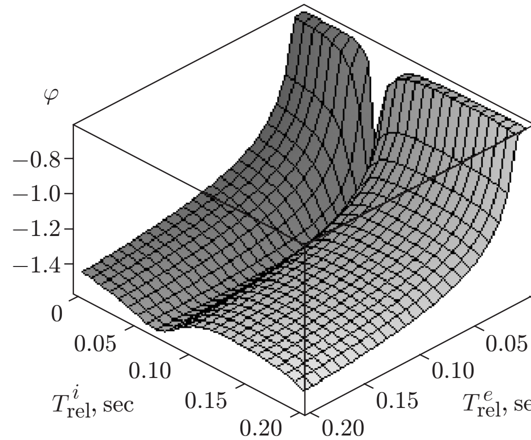

Fig. 2

The conjugation conditions for P acquire the form

$$\{(-\hat{\alpha}^+ p^+ + \hat{\alpha}^- p^-) - (\rho^0 - 1)x_3[A - i\omega u]\}\boldsymbol{n}$$

$$+ 2(\hat{\alpha}^+ \mu^0 D(\boldsymbol{v}^+) \cdot \boldsymbol{n} - \hat{\alpha}^- D(\boldsymbol{v}^-) \cdot \boldsymbol{n}) = -2i\sigma/(ag_0\omega) \cdot \boldsymbol{n},$$

where $\rho^0 = \rho^+/\rho^-$, $\nu^0 = \nu^+/\nu^-$, and $\mu^0 = \rho^0\nu^0$.

Introducing a spherical coordinate system and applying a procedure similar to solving the problem with a monotonic function $g(t)$, we obtain the expression for velocity:

$$u(t) = \mathrm{Re}\, U(t), \qquad U(t) = \frac{(\rho^0 - 1)A\, \mathrm{e}^{i\omega t}}{(1/2 + \rho^0)i\omega + \alpha^- \hat{B}(i\omega)}. \tag{20}$$

Passing to the limit in terms of density and viscosity, we can easily obtain the velocity of a gas bubble or a solid sphere from Eq. (20).

It seems of interest to analyze the dependence of the droplet velocity and the shift of the phase of oscillations on parameters of the relaxation time $T_{\rm rel}$ of the external medium and droplet. To construct these dependences, we theoretically vary the properties or types of fluids so that the relaxation time changes and the other properties remain intact. As a "reference point," we consider a 0.5% solution of polyacrylamide. This solution is a typical representative of the Maxwellian fluid with a rather long relaxation time (from 0.1 to 5 sec). Its dynamic viscosity is $\mu = 5$ P.

Figures 1 and 2 show the amplitudes of velocity $u = |U(t)|$ and the shift of the phase of oscillations of the droplet $\varphi = \arg U(t)$, respectively (the velocity is measured in hundredths of cm/sec). The frequency of oscillations of the driving force is $\omega = 500$ Hz and the amplitude of acceleration is $g = 5$ cm/sec^2, which is almost 200 times smaller than the acceleration of gravity on the Earth's surface; the ratio of densities of the internal and external fluids is $\rho^0 = 2$, and the ratio of dynamic viscosities is $\mu^0 = 4$. It is seen that the dependence of the examined quantities on the relaxation time of the external fluid is more pronounced. In addition, as the relaxation time of the external fluid $T^e_{\rm rel}$ increases, the influence of the relaxation time of the internal fluid $T^i_{\rm rel}$ on the motion pattern decreases. An interesting and unexpected result was the presence of peaks and valleys in all graphs. At the moment, this fact has not been adequately explained, but apparently, it is a consequence of the resonance of intrinsic oscillations of the droplet and oscillations of the driving force.

Now we consider the relaxation time of the external fluid $T^e_{\rm rel}$ and the frequency ω as parameters. We fix the value $T^i_{\rm rel} = 0.1$ sec.

In this case (Fig. 3), if the oscillation frequency is rather low (about 100 Hz), the droplet velocity amplitude increases with increasing relaxation time of the medium $T^e_{\rm rel}$. As the frequency increases to 500 Hz, this dependence is no longer valid, and the value of $T^e_{\rm rel}$ has almost no effect on the droplet velocity. For $\omega \approx 100$ Hz, the shift of the phase of oscillations is close to zero (Fig. 4). For $\omega = 300$ Hz and higher, vice versa, the phase shift reaches $\pi/2$.

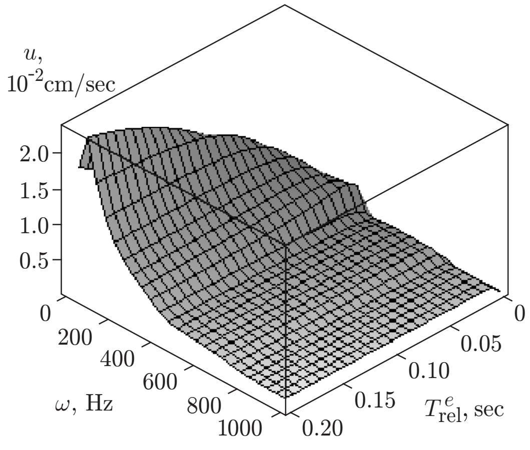

Fig. 3

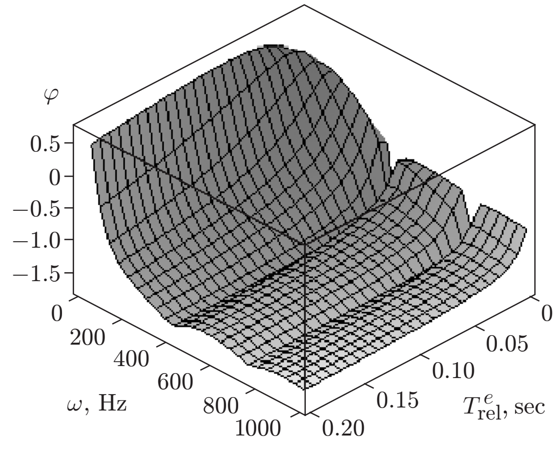

Fig. 4

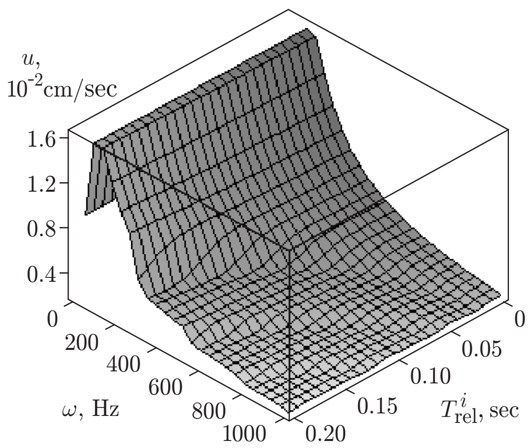

Fig. 5

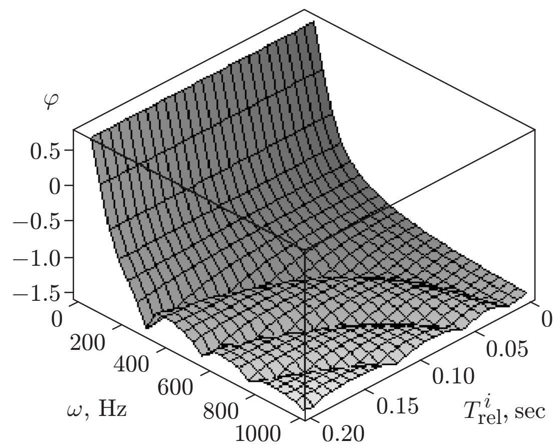

Fig. 6

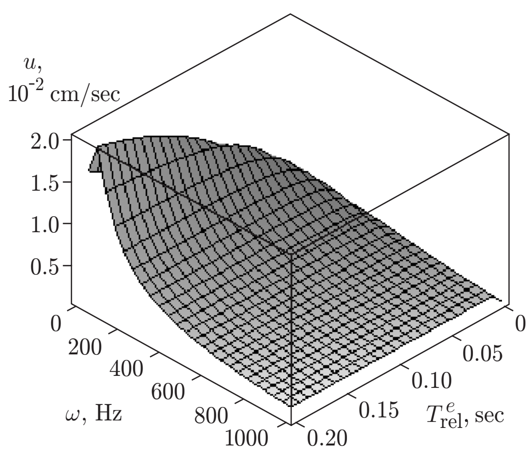

Fig. 7

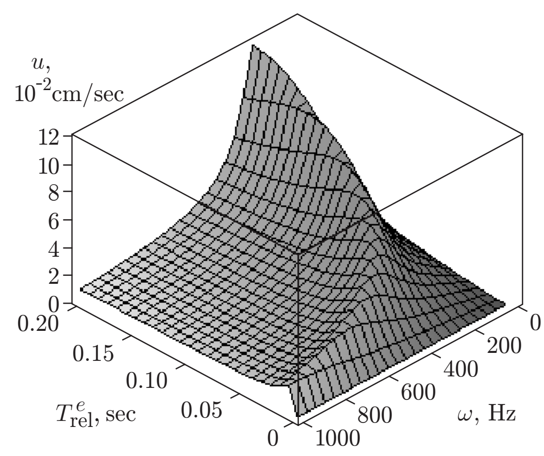

Fig. 8

Note, both plots become less "hump-backed" with increasing $T^e_{\rm rel}$, which indicates that the relative influence of $T^i_{\rm rel}$ becomes less pronounced.

Figures 5 and 6 show the dependences of the examined quantities on the relaxation time of the internal fluid and on the frequency of oscillations of the external force ($T^e_{\rm rel} = 0.1$ sec). Obviously, the relaxation time of the external medium has a greater effect on the flow pattern.

Figures 7 and 8 show the velocity amplitude as a function of the relaxation time of the external medium and on the frequency of oscillations of the driving force for the cases of motion of a solid sphere and a gas bubble, respectively.

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